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Open Newton-Cotes Quadrature with Midpoint Derivative

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ABSTRACT

In this paper, as a new approach arithmetic mean of end points is applied at the error term derivative in the existing ONC formulas. The new error terms are also derived by applying the method of precision. This arithmetic mean derivative based ONC formulas increase the two orders of precision than the previous ONC formulas and the left out statistical means are the scope for future studies in the ONC formulas.

Keywords- *ONC formulas, Newton cotes formulas*

I. INTRODUCTION

The definite integral can also be approximated by using another type of Newton-Cotes formulas that are open Newton-Cotes quadrature formulas. In the earlier work, Weijiing Zhao and Hongxing Li applied the arithmetic mean of end points at the error term derivative and it is included as an additional term to the existing CNC formulas [1]. These improved formulas give better results for the approximation of definite integrals. As a continuation of that, the existing CNC formulas are modified in the previous chapters by applying the Geometric mean, Heronian mean, Harmonic mean, Contra-harmonic mean, Centroidal mean and Root mean square of end points at the error term derivative. These mean derivative-based CNC formulas give improved results than the existing results.

In the ONC formulas, the end points of the interval are excluded in the function evaluation. The general form of the ONC quadrature is

$$\int_a^b f(x)dx = \int_{x_{-1}}^{x_{n+1}} f(x)dx \approx \sum_{i=0}^{n} w_i f\,(x_i) \qquad (1.1)$$

Where $w_i = (n+1)$ weights $w_0, w_1, \dots, w_n$

$x_i = (n+1)$intermediate points $x_0, x_1, \dots, x_n$

Usually, these weights can be determined in several different ways [2,3,4,5,6]. One of the methods is to derive a set of ONC formulas by using Lagrange's interpolation polynomials.

If n is even

$$\int_a^b f(x)dx = \int_{x_{-1}}^{x_{n+1}} f(x)dx$$
$$\approx \sum_{i=0}^{n} w_i f\,(x_i) + \frac{h^{n+3} f^{(n+2)}(\xi)}{(n+2)!}\int_{-1}^{n+1} t^2(t-1)\dots(t-n)dt, \qquad (1.2)$$

and if n is odd

$$\int_a^b f(x)dx = \int_{x_{-1}}^{x_{n+1}} f(x)dx$$
$$\approx \sum_{i=0}^{n} w_i f\,(x_i) + \frac{h^{n+2} f^{(n+1)}(\xi)}{(n+1)!}\int_{-1}^{n+1} t^2(t-1)\dots(t-n)dt, \qquad (1.3)$$

Where $h = \frac{b-a}{n+2}, \xi \epsilon (a,b)$

And $w_i = \int_a^b \prod_{\substack{j=0 \\ j\neq i}}^{n} \frac{x-x_j}{x_i-x_j}dx.$

Therefore, the first four ONC formulas for n = 0, 1, 2, 3 are given as follows:

If n = 0;

$\int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f^{(2)}(\xi),$

where $\xi\ \epsilon(a,b)$ (1.4)

If n =1;

$\int_a^b f(x)dx = \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] + \frac{(b-a)^3}{36} f^{(2)}(\xi),$
where $\xi \, \epsilon(a, b)$ (1.5)
If n=2;
$\int_a^b f(x)dx = \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] + \frac{14}{45}\left(\frac{b-a^5}{4}\right) f^{(4)}(\xi),$
where $\xi \, \epsilon(a, b)$ (1.6)
If n=3;
$$\int_a^b f(x)dx = \frac{b-a}{24}\left[11f\left(\frac{4a+b}{5}\right) + f\left(\frac{3a+2b}{5}\right) + f\left(\frac{2a+3b}{5}\right) + 11f\left(\frac{a+4b}{5}\right)\right] + \frac{95}{144}\left(\frac{b-a^5}{5}\right) f^{(4)}(\xi),$$
where $\xi \, \epsilon(a, b)$ (1.7)
In the next section, a selective review of the existing development of ONC formulas is described.

1.1.1 Existing results on ONC formulas

There are a number of mathematicians who made major contributions to not only the CNC formulas but also to the ONC formulas in different dimensions.

Symplectic integrators

In 1996, Zhu et al. derived [7] the open Newton-Cotes differential methods from the ONC formulas.

$$\int_a^b f(x)dx = \int_{x_{-1}}^{x_{n+1}} y(x)dx \approx \sum_{i=0}^{n} a_i y\,(x_i) + (h^{n+1}), \qquad (1.8)$$

and converted the above equation into the multilayer symplectic integrators for solving the linear Hamiltonian's equation of motion. This integrator has some ambiguous expression in deriving the accuracy and it was resolved by Chiou and Wu [8] in 1997. Following them, Vanden Berghe and Van Deale developed the exponentially fitted ONC formulas [9] of order two, four and six for solving the Hamiltonian system. In 2011, Simos developed the trigonometrically-fitted NC type multilayer symplectic integrators for solving the Schr ̈dinger equation [10]. Numerical examples are also demonstrated for the effectiveness of the proposed differential methods.

Fredholm integral

In 2011, Ibraheem introduced a new method for approximating the linear fredholm integral equation of second kind by the ONC formulas [11].

Improved ONC formulas

In 2006, Dehghan et al. improved the precision degree of the ONC formulas by considering the integral bounds as two additional unknowns [12]. Therefore, the ONC formulas became,

$$\int_a^b f(x)dx = \frac{b-a}{n+2} \sum_{i=0}^{n} B_k^{(n)}\,(a + (k+1)h) \qquad (1.9)$$

Where,

$$B_k^{(n)} = \frac{(-1)^{n-k}}{k!(n-k)!} \int_{-1}^{n+1} t(t-1)(t-2)\ldots(t-k+1)(t-k-1)\ldots(t-n)dt, \qquad (1.10)$$

The new improved integration formulas are more efficient and they increase the order of precision by two than the existing ONC formulas. The new error terms are also derived by using the concept of precision.
In the next section, a survey of derivative-based improvement of ONC formulas is discussed.

1.1.2 Existing results on ONC-based on Derivatives

In this section, the derivative-based formulas which include the derivative as an additional parameter for the existing ONC formulas are described.

Derivative-based Midpoint formula

In 2013, Burg introduced a modified midpoint formula which has the midpoint for the function evaluation and endpoints for odd derivatives [13]. The eighth-order midpoint formula can be defined as

$$\int_a^b f(x)dx = 2h \int_{x_{-1}}^{\frac{n}{2}} f(x_{2i-1}) - \frac{h^2}{6}\left(f^{(1)}(a) - f^{(1)}(b)\right) + \frac{7h^3}{360}\left(f^{(1)}(a) - f^{(1)}(b)\right) - \frac{31h^6}{15120}\left(f^{(5)}(a) - f^{(5)}(b)\right) - \frac{127h^8}{604800}(b-a)f^{(8)}(\xi),$$

where $\xi \, \epsilon(a, b)$ (1.11)

This quadrature formula has two first, third and fifth derivatives and N/2 function evaluations. Also, the method of precision is used to derive those derivatives. This improved formula gives accurate results than the existing Midpoint formula.

Derivative-based ONC formulas

In 2014, Fiza Zafar et al. concentrated on improving the ONC formulas with less number of points and higher order of accuracy when compared with the existing ONC formulas. Therefore, they applied the first derivative evaluation at all interior points, all points including end points, interior end points and only at end points [14]. Finally, these derivatives were included as an additional parameter to the existing ONC formulas and the error terms were derived by using the concept of precision.

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The ONC formulas which used the function evaluation and first derivative of an integrand at all interior points $x_i \in (a,b)$ is defined as

$$\int_{x_{-1}}^{x_{n+1}} f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) + \sum_{i=0}^{n} u_i f'(x_i)h, \qquad (1.12)$$

The ONC formulas which used the function evaluation of an integrand at all interior points $x_i \in (a,b)$ and first derivative at all points including end points $x_i \in [a,b]$ is defined as

$$\int_{x_{-1}}^{x_{n+1}} f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) + \sum_{i=-1}^{n+1} u_i f'(x_i)h, \qquad (1.13)$$

The ONC formulas which used the function evaluation of an integrand at all interior points $x_i \in (a,b)$ and first derivative at interior end points x_0 and x_n is defined as

$$\int_{x_{-1}}^{x_{n+1}} f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) + c_0 f'(x_0)h - c_1 f'(x_n)h, \quad (1.14)$$

The ONC formulas which use the function evaluation of an integrand at all interior points $x_i \in (a,b)$ and first derivative at end points and is defined as

$$\int_{x_{-1}}^{x_{n+1}} f(x)dx \approx \sum_{i=0}^{n} w_i f(x_i) + u_0 f'(x_{-1})h - u_1 f'(x_{n+1})h, \quad (1.14)$$

where w_i = weights for the function
u_i = weights for the derivative
n = number of subintervals

Finally, these derivative based ONC formulas produce more accurate numerical results than the existing ONC formulas.

In the next section, as a new approach the arithmetic mean of end points or midpoint is applied at the derivative of error term and is included as an additional function to the existing first four ONC formulas. These Open Newton-Cotes quadrature with midpoint derivative (ONC-MD) formula gives better results than the existing ONC formulas.

II. ONC-MD FORMULAS

A new ONC-MD formula is explained below which gives higher precision than the classical Newton-Cotes formulas. The midpoint derivative technique is applied only to the first four ONC formulas for n = 0, 1, 2, 3.

Theorem 1.1

The first ONC-MD formula for n = 0 is

$$\int_a^b f(x)dx \approx (b-a) f\left(\frac{a+b}{2}\right) + \left(\frac{(b-a)^3}{24}\right) f^{(2)}\left(\frac{a+b}{2}\right) \qquad (1.16)$$

The precision of this method is 3.

Proof:

The formula (1.4) has the degree of precision 1. Now the formula (1.16) is exact for $f(x) = x^2, x^3$

The exact value of $\int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3)$;

(1.16)$\Longrightarrow (b-a)\left(\frac{a+b}{2}\right)^2 + \frac{2(b-a)^3}{24} = \frac{1}{3}(b^3 - a^3)$.

The exact value of $\int_a^b x^3 dx = \frac{1}{4}(b^4 - a^4)$;

(1.16)$\Longrightarrow (b-a)\left(\frac{a+b}{2}\right)^3 + \frac{6(b-a)^3}{24}\left(\frac{a+b}{2}\right) = \frac{1}{4}(b^4 - a^4)$.

Therefore, the precision of the first ONC-MD formula is 3.

Theorem 1.2

The second ONC-MD formula for n = 1 is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] + \left(\frac{(b-a)^3}{36}\right) f^{(2)}\left(\frac{a+b}{2}\right) \qquad (1.17)$$

The precision of this method is 3.

Proof:

The formula (1.5) has the degree of precision 1. Now the formula (1.16) is exact for $f(x) = x^2, x^3$

The exact value of $\int_a^b x^2 dx = \frac{1}{3}(b^3 - a^3)$;

(1.17)$\Longrightarrow \left(\frac{b-a}{2}\right)\left[\left(\frac{2a+b}{3}\right)^2 + \left(\frac{a+2b}{3}\right)^2\right] + \frac{2(b-a)^3}{36} = \frac{1}{3}(b^3 - a^3)$.

The exact value of $\int_a^b x^3 dx = \frac{1}{4}(b^4 - a^4)$;

(1.16)$\Longrightarrow \left(\frac{b-a}{2}\right)\left[\left(\frac{2a+b}{3}\right)^3 + \left(\frac{a+2b}{3}\right)^3\right] + \frac{6(b-a)^3}{36}\left(\frac{a+b}{2}\right)$

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$$= \frac{1}{4}(b^4 - a^4).$$

Therefore, the precision of the second ONC-MD formula is 3.

Theorem 1.3

The third ONC-MD formula for n = 2 is

$$\int_a^b f(x)dx \approx \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] + \frac{14}{45}\left(\frac{(b-a)^5}{4}\right) f^{(4)}\left(\frac{a+b}{2}\right) \quad (1.18)$$

The precision of this method is 5.

Proof:

The formula (1.6) has the degree of precision 1. Now the formula (1.18) is exact for $f(x) = x^4, x^5$

The exact value of $\int_a^b x^4 dx = \frac{1}{5}(b^5 - a^5)$;

$$(1.18) \Longrightarrow \left(\frac{b-a}{3}\right)\left[2\left(\frac{3a+b}{4}\right)^4 - \left(\frac{a+b}{2}\right)^4 + 2\left(\frac{a+3b}{4}\right)^4\right] + \frac{336}{45}\left(\frac{b-a}{4}\right)^5$$
$$= \frac{1}{5}(b^5 - a^5).$$

The exact value of $\int_a^b x^5 dx = \frac{1}{6}(b^6 - a^6)$;

$$(1.18) \Longrightarrow \left(\frac{b-a}{3}\right)\left[2\left(\frac{3a+b}{4}\right)^5 - \left(\frac{a+b}{2}\right)^5 + 2\left(\frac{a+3b}{4}\right)^5\right] + \frac{1680}{45}\left(\frac{b-a}{4}\right)^5\left(\frac{a+b}{2}\right)$$
$$= \frac{1}{6}(b^6 - a^6).$$

Therefore, the precision of the third ONC-MD formula is 5.

Theorem 1.4

The fourth ONC-MD formula for n = 3 is

$$\int_a^b f(x)dx \approx \frac{b-a}{24}\left[11f\left(\frac{4a+b}{5}\right) + f\left(\frac{3a+2b}{5}\right) + f\left(\frac{2a+3b}{5}\right) + 11f\left(\frac{a+4b}{5}\right)\right] + \frac{95}{144}\left(\frac{(b-a)^5}{4}\right) f^{(4)}\left(\frac{a+b}{2}\right) \quad (1.19)$$

The precision of this method is 5.

Proof:

The formula (1.7) has the degree of precision 3. Now the formula (1.19) is exact for $f(x) = x^4, x^5$

The exact value of $\int_a^b x^4 dx = \frac{1}{5}(b^5 - a^5)$;

$$(1.19) \Longrightarrow \left(\frac{b-a}{24}\right)\left[11\left(\frac{4a+b}{5}\right)^4 + \left(\frac{3a+2b}{5}\right)^4 + \left(\frac{2a+3b}{5}\right)^4 + 11\left(\frac{a+4b}{5}\right)^4\right] + \frac{2280}{144}\left(\frac{b-a}{5}\right)^5$$
$$= \frac{1}{5}(b^5 - a^5).$$

The exact value of $\int_a^b x^5 dx = \frac{1}{6}(b^6 - a^6)$;

$$(5.19) \Longrightarrow \left(\frac{b-a}{24}\right)\left[11\left(\frac{4a+b}{5}\right)^5 + \left(\frac{3a+2b}{5}\right)^5 + \left(\frac{2a+3b}{5}\right)^5 + 11\left(\frac{a+4b}{5}\right)^5\right]$$
$$+ \frac{11400}{144}\left(\frac{b-a}{5}\right)^5\left(\frac{a+b}{2}\right) = \frac{1}{6}(b^6 - a^6).$$

Therefore, the precision of the fourth ONC-MD formula is 5.

III. THE ERROR TERMS OF ONC-MD FORMULAS

The Error terms of ONC - MD formulas are given below. These Error terms are obtained by using the difference between the exact value and the quadrature formulas for the monomial $\frac{x^{p+1}}{(p+1)!}$.

Theorem 1.5

The error term of first ONC - MD formula for n = 0 is

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right) + \left(\frac{(b-a)^3}{24}\right) f^{(2)}\left(\frac{a+b}{2}\right) + \frac{(b-a)^5}{1920} f^{(4)}(\xi), \quad (1.20)$$

where $\xi \epsilon$ (a, b). This is the fourth order accurate with the error term

$$E_{25}[f] = -\frac{(b-a)^5}{1920} f^{(4)}(\xi).$$

Proof:

The exact value of $\frac{1}{4!}\int_a^b x^4 dx = \frac{1}{120}(b^5 - a^5)$;

$$(b-a)f\left(\frac{a+b}{2}\right) + \left(\frac{(b-a)^3}{24}\right) f^{(2)}\left(\frac{a+b}{2}\right)$$
$$= \frac{b-a}{4!.16}(3a^4 + 4a^3b + 2a^2b^2 + 4ab^3 + 3b^4),$$

Therefore,

$$E_{25}[f] = -\frac{1}{120}(b^5 - a^5) - \frac{b-a}{4!.16}(3a^4 + 4a^3b + 2a^2b^2 + 4ab^3 + 3b^4)$$
$$= \frac{(b-a)^5}{1920}.$$

Therefore, the error term is

$$E_{25}[f] = \frac{(b-a)^5}{1920} f^{(4)}(\xi).$$

Theorem 1.6

The error term of the second ONC-MD formula for n = 1is

$$\int_a^b f(x)dx \approx \frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] + \frac{(b-a)^3}{36} f^{(2)}\left(\frac{a+b}{2}\right) + \frac{19(b-a)^5}{38880} f^{(4)}(\xi), \qquad (1.21)$$

where $\xi \epsilon$ (a, b). This is the fourth order accurate with the error term

$$E_{26}[f] = -\frac{19(b-a)^5}{38880} f^{(4)}(\xi).$$

Proof:

The exact value of $\frac{1}{4!}\int_a^b x^4 dx = \frac{1}{120}(b^5 - a^5)$;

$$\frac{b-a}{2}\left[f\left(\frac{2a+b}{3}\right) + f\left(\frac{a+2b}{3}\right)\right] + \frac{(b-a)^3}{36} f^{(2)}\left(\frac{a+b}{2}\right)$$
$$= \frac{b-a}{7776}(61a^4 + 80a^3b + 42a^2b^2 + 80ab^3 + 61b^4),$$

Therefore,

$$E_{26}[f] = \frac{1}{120}(b^5 - a^5) - \frac{b-a}{7776}(61a^4 + 80a^3b + 42a^2b^2 + 80ab^3 + 61b^4)$$
$$= \frac{19(b-a)^5}{38880}.$$

Therefore, the error term is

$$E_{26}[f] = \frac{19(b-a)^5}{38880} f^{(4)}(\xi).$$

Theorem 1.7

The error term of the third ONC-MD formula for n = 2 is

$$\int_a^b f(x)dx \approx \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] + \frac{14}{45}\left(\frac{b-a}{4}\right)^5 f^{(4)}\left(\frac{a+b}{2}\right) + \frac{41(b-a)^7}{15482880} f^{(6)}(\xi), \qquad (1.22)$$

where $\xi \epsilon$ (a, b). This is the fourth order accurate with the error term

$$E_{27}[f] = \frac{41(b-a)^5}{15482880} f^{(6)}(\xi).$$

Proof:

The exact value of $\frac{1}{6!}\int_a^b x^6 dx = \frac{1}{5040}(b^7 - a^7)$;

$$\frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] + \frac{14}{45}\left(\frac{b-a}{4}\right)^5 f^{(4)}\left(\frac{a+b}{2}\right)$$
$$= \frac{b-a}{3072.6!}\begin{pmatrix} 4336a^6 + 474a^5b + 351a^2b^2 + 556a^3b^3 \\ +351a^2b^4 + 474ab^5 + 433b^6 \end{pmatrix},$$

Therefore,

$$E_{27}[f] = \frac{(b^7 - a^7)}{5040} - \frac{b-a}{3072.6!}\begin{pmatrix} 4336a^6 + 474a^5b + 351a^2b^2 + 556a^3b^3 \\ +351a^2b^4 + 474ab^5 + 433b^6 \end{pmatrix}$$
$$= \frac{41(b-a)^7}{15482880}.$$

Therefore, the error term is

$$E_{27}[f] = \frac{41(b-a)^7}{15482880} f^{(6)}(\xi).$$

Theorem 1.8

The error term of the fourth ONC-MD formula for n = 3 is

$$\int_a^b f(x)dx \approx \frac{b-a}{24}\left[11f\left(\frac{4a+b}{5}\right) + f\left(\frac{3a+2b}{5}\right) + f\left(\frac{2a+3b}{5}\right) + 11f\left(\frac{a+4b}{5}\right)\right] + \frac{95}{144}\left(\frac{b-a}{5}\right)^5 f^{(4)}\left(\frac{a+b}{2}\right) + \frac{821(b-a)^7}{168.5^5.6!} f^{(6)}(\xi), \quad (1.23)$$

where $\xi \epsilon$ (a, b). This is the fourth order accurate with the error term

$$E_{28}[f] = \frac{821(b-a)^7}{168.5^5.6!} f^{(6)}(\xi).$$

Proof:

The exact value of $\frac{1}{6!}\int_a^b x^6 dx = \frac{1}{5040}(b^7 - a^7)$;

$$\frac{b-a}{24}\left[11f\left(\frac{4a+b}{5}\right) + f\left(\frac{3a+2b}{5}\right) + f\left(\frac{2a+3b}{5}\right) + 11f\left(\frac{a+4b}{5}\right)\right] + \frac{95}{144}\left(\frac{b-a}{5}\right)^5 f^{(4)}\left(\frac{a+b}{2}\right)$$
$$= \frac{b-a}{375000.6!}\begin{pmatrix} 52985a^6 + 57090a^5b + 44775a^4b^2 + 65300a^3b^3 \\ +44775a^2b^4 + 57090ab^5 + 52985b^6 \end{pmatrix},$$

Therefore,

$$E_{28}[f]=\frac{(b^7-a^7)}{5040}-\frac{b-a}{375000.6!}\begin{pmatrix}52985a^6+57090a^5b+44775a^4b^2+65300a^3b^3\\ +44775a^2b^4+57090ab^5+52985b^6\end{pmatrix}$$
$$=\frac{821(b-a)^7}{168.5^5.6!}.$$

Therefore, the error term is

$$E_{28}[f]=\frac{821(b-a)^7}{168.5^5.6!}f^{(6)}(\xi).$$

IV. SUMMARY

The following Table 1.1 gives a summary of precision, the order and the error terms of the existing ONC formulas and the Midpoint derivative-based ONC formulas.

Table 1.1: The Error terms of ONC and ONC-MD formulas

Formulas	ONC			ONC-MD		
	Precision	Order	Error terms	Precision	Order	Error terms
n = 0	1	3	$\frac{(b-a)^3}{24}f^{(2)}(\xi)$	3	5	$\frac{(b-a)^5}{1920}f^{(4)}(\xi)$
n = 1	1	3	$\frac{(b-a)^3}{36}f^{(2)}(\xi)$	3	5	$\frac{19(b-a)^5}{38880}f^{(4)}(\xi)$
n = 2	3	5	$\frac{14}{45}\left(\frac{b-a}{4}\right)^5 f^{(4)}(\xi)$	5	7	$\frac{41(b-a)^7}{15482880}f^{(6)}(\xi)$
n = 3	3	5	$\frac{95}{144}\left(\frac{b-a}{5}\right)^5 f^{(4)}(\xi)$	5	7	$\frac{821(b-a)^7}{168.5^5.6!}f^{(6)}(\xi)$

The comparison indicates that the proposed midpoint derivative-based ONC formulas increase the two orders of precision than the existing formulas.

V. NUMERICAL EXAMPLES

An approximate value of the following examples using the ONC-MD formulas is determined and presented. To demonstrate the accuracy of the results, we evaluate the examples 1.1 – 1.3 and the comparison of results is shown in Tables 1.2 – 1.4.

Example 1.1

Solve $\int_0^2 e^x dx$ and compare the solutions with the ONC and ONC-MD formulas.

Solution:

The exact value of $\int_0^2 e^x dx$ = 6.389056989.

Table 1.2: Comparison of ONC and ONC-MD formulas - Example 1.1

Formulas	ONC		ONC-MD	
	Approximate Value	Error	Approximate Value	Error
n = 0	5.436563657	0.952492441	6.342657600	0.046398498
n = 1	5.741401936	0.647654162	6.345464564	0.043515340
n = 2	6.361692569	0.027363529	6.388120309	0.000935789
n = 3	6.369923849	0.019132249	6.388287352	0.000768746

Example 1.2

Solve $\int_0^1 \frac{dx}{1+x}$ and compare the solutions with the ONC and ONC-MD formulas.

Solution:

The exact value of $\int_0^1 \frac{dx}{1+x} = 0.69314718$.

Table 1.3: Comparison of ONC and ONC-MD formulas - Example 1.2

Formulas	ONC		ONC-MD	
	Approximate Value	Error	Approximate Value	Error
n = 0	0.666666666	0.026480520	0.691358018	0.001789162
n = 1	0.675000000	0.018147180	0.691460905	0.001686275
n = 2	0.692063492	0.001083688	0.693023711	0.000123469
n = 3	0.692377645	0.000769535	0.693044860	0.000102320

Example 1.3

Solve $\int_0^1 (1+x^4)dx$ and compare the solutions with the ONC and ONC-MD formulas.

Solution:

The exact value of $\int_0^1 (1+x^4)dx = 1.2$.

Table 1.4: Comparison of ONC and ONC-MD formulas - Example 1.3

Formulas	ONC		ONC - MD	
	Approximate Value	Error	Approximate Value	Error
n = 0	1.062500000	0.137500000	1.187500000	0.012500000
n = 1	1.104938272	0.095061728	1.188271605	0.011728395
n = 2	1.192708333	0.007291666	1.200000000	0.000000000
n = 3	1.194933333	0.005066666	1.200000000	0.000000000

VI. CONCLUSION

In this paper, as a first approach only the arithmetic mean of end points is applied at the derivative of error term to improve the accuracy of the existing ONC formulas. The remaining statistical means are left for future scope. From the results presented in Tables 1.2 - 1.4, it is observed that the ONC-MD formulas give more accuracy than the standard ONC.

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